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On gradient-induced instabilities

R H C Newton and D M Short

School of Electronic Engineering Science, University College of North Wales, Dean Street, Bangor, Caerns, UK

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Abstract. It is shown that the unstable behaviour of a single-carrier stream in the presence of a carrier density gradient is due to wave interaction between an active mode and a passive mode: modes which do not couple when the carrier density is uniform.

1. Introduction

Of the many known instabilities in a plasma system most are explained in terms of an interaction between different components of the system. For example unstable behaviour is expected when electron streams and plasmas interact. Again collision-induced instabilities in solid-state plasmas require the presence of holes in the case of longitudinal waves. An apparent exception occurs in the case of transverse waves when a collision-induced instability can exist (theoretically) when no holes are present. However, in this case the lattice itself plays the rôle of the second component of the system (see eg Steele and Vural 1969).

Recently a class of instability, which we shall call *gradient-induced*, has been proposed. One example which depends on temperature gradient as well as carrier density gradient has already been reported (Boyd *et al* 1972). A feature of this class of instability is that, as in the case of transverse waves, holes do not play an essential part (although they modify the results), but in addition the waves can be entirely electromechanical, and no propagating properties of the lattice need to be assumed.

This raises the question: given that there is only a single carrier stream present, how can there be growth of wave amplitude?

This paper attempts to answer this question by simplifying the problem to that of a plasma system with no temperature gradient (and hence no Nernst effect or thermal force), but with a density gradient such as could be induced in a material by nonuniform doping or by exposing it to nonuniform radiation. Crossed electric and magnetic fields are assumed since experiments to date show unstable behaviour only when there is a significant transverse component of magnetic field (Boyd *et al* 1972). Furthermore, since Landau damping is inhibited under these assumptions we can have greater confidence in the hydrodynamical approach used here.

To summarize our results we mention that in § 6 Briggs' criteria are formally applied to a dispersion relation derived under the quasistatic approximation. It is found that, although there is no absolute instability, a convective instability is present. In § 7 the equations are reduced to normal mode form. In the limit of zero temperature, zero magnetic field and no collisions these normal modes become the usual Hahn–Ramo 'fast' and 'slow' space-charge waves.

These modes are uncoupled, but when there is a nonzero density gradient they become strongly active-coupled as shown in §§ 7 and 9.

2. Basic equations

The equations of continuity and force are, in rationalized MKS units,

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \rho \boldsymbol{v}$$

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \frac{e}{m}(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B}) - \nu\boldsymbol{v} - \frac{kT}{m}\frac{\nabla\rho}{\rho},$$

where $\nu\boldsymbol{v}$ is the frictional force between the particle concerned and the environment (eg the lattice and other particles).

In a solid-state plasma of holes and electrons for example e, ν, m, T, ρ and $\boldsymbol{v}$ will be different for the two classes of carrier, in general. The electric field $\boldsymbol{E}$ and magnetic flux density $\boldsymbol{B}$ will refer to the plasma as a whole.

There will be a heat flux needed to maintain the system at uniform temperature, but we will not be concerned with this in § 5. Accordingly the heat balance equation will not be required.

3. First-order perturbations

In what follows a suffix 0 will indicate a steady-state (zero-order) variable, and a suffix 1 will indicate a time-varying (first-order) variable. Then $\rho = \rho_0 + \rho_1$, $\boldsymbol{v} = \boldsymbol{v}_0 + \boldsymbol{v}_1$ etc.

We choose axes such that $\boldsymbol{B} = B\boldsymbol{k}$, where $\boldsymbol{k} = (0, 0, 1)$ is the unit vector along the z axis. We suppose also that the first order terms vary as $\exp\{\mathrm{i}(\omega t - \boldsymbol{\beta}\cdot\boldsymbol{r})\}$ where $\boldsymbol{r} = (r_1, r_2, r_3) = (x, y, z)$ is the radius vector from the origin.

If wavelengths are much smaller than density decay lengths we may make the 'local approximation' by assuming $\boldsymbol{\beta}$ independent of $\boldsymbol{r}$. When this is the case the continuity equation yields

$$0 = \nabla \cdot \rho_0 \boldsymbol{v}_0 = \rho_0 \nabla \cdot \boldsymbol{v}_0 + \boldsymbol{v}_0 \cdot \nabla \rho_0$$

or

$$\nabla \cdot \boldsymbol{v}_0 = -\boldsymbol{v}_0 \cdot \boldsymbol{\delta} \tag{3.1}$$

where

$$\boldsymbol{\delta} = \frac{\nabla \rho_0}{\rho_0},$$

and

$$\{\omega - \boldsymbol{v}_0 \cdot (\boldsymbol{\beta} - \mathrm{i}\boldsymbol{\delta})\}\frac{\rho_1}{\rho_0} = (\boldsymbol{\beta} + \mathrm{i}\boldsymbol{\delta}) \cdot \boldsymbol{v}_1 . \tag{3.2}$$

The zero-order force equation is

$$(\boldsymbol{v}_0 \cdot \nabla)\boldsymbol{v}_0 = \frac{e}{m}\boldsymbol{E}_0 + \omega_c \boldsymbol{v}_0 \times \boldsymbol{k} - \nu\boldsymbol{v}_0 - \frac{kT}{m}\frac{\nabla\rho_0}{\rho_0},$$

and subtracting this from the force equation yields the first-order force equation

$$[\{\mathrm{i}(\omega-\boldsymbol{v}_0\,.\,\boldsymbol{\beta})+\nu\}\,\|U\|-\omega_\mathrm{c}\|\boldsymbol{M}\|+\|\boldsymbol{\theta}\|]\,.\,\boldsymbol{v}_1=\frac{e}{m}\boldsymbol{E}_1+\mathrm{i}\frac{kT}{m}(\boldsymbol{\beta}-\mathrm{i}\boldsymbol{\delta})\frac{\rho_1}{\rho_0}.$$

The two tensors $\|U\|$, $\|\boldsymbol{M}\|$, are

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\quad\text{and}\quad\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix}$$

respectively, and an element of the tensor $\|\boldsymbol{\theta}\|$ is $\theta_{lk}=\partial v_{0l}/\partial r_k$†. If ρ_1/ρ_0 is eliminated from this equation by means of (3.2) the final relation between $\boldsymbol{v}_1$ and $\boldsymbol{E}_1$ is obtained. This can be written as

$$[\{\mathrm{i}(\omega-\boldsymbol{v}_0\,.\,\boldsymbol{\beta})+\nu\}\,\|U\|-\omega_\mathrm{c}\|\boldsymbol{M}\|+\|\boldsymbol{\theta}\|-\mathrm{i}\|\boldsymbol{g}\|]\,.\,\boldsymbol{v}_1=\frac{e}{m}\boldsymbol{E}_1,$$

where

$$g_{kl}=\frac{kT}{m}\frac{(\beta_k-\mathrm{i}\delta_k)(\beta_l+\mathrm{i}\delta_l)}{\omega-\boldsymbol{v}_0\,.\,\boldsymbol{\beta}+\mathrm{i}\boldsymbol{v}_0\,.\,\boldsymbol{\delta}}. \tag{3.3}$$

4. The dispersion equation for a simple carrier system (no holes) in the quasistatic approximation

The procedure is first to invert the tensor in (3.3) which gives

$$\boldsymbol{v}_1=\frac{e}{m}\|\boldsymbol{\xi}\|\,.\,\boldsymbol{E}_1, \tag{3.4}$$

the tensor $(e/m)\|\boldsymbol{\xi}\|$ being the (dynamic) mobility. Then to calculate the first-order current density

$$\begin{aligned}\boldsymbol{j}&=\rho_0\boldsymbol{v}_1+\rho_1\boldsymbol{v}_0\\&=\rho_0\left(\frac{e}{m}\|\boldsymbol{\xi}\|\,.\,\boldsymbol{E}_1+\boldsymbol{v}_0\frac{(\boldsymbol{\beta}+\mathrm{i}\boldsymbol{\delta})\,.\,\boldsymbol{v}_1}{\omega-\boldsymbol{v}_0\,.\,\boldsymbol{\beta}+\mathrm{i}\boldsymbol{v}_0\,.\,\boldsymbol{\delta}}\right)\qquad\text{(from (3.2))}\\&=\epsilon\epsilon_0\omega_\mathrm{p}^2(\|\boldsymbol{\xi}\|+\|\boldsymbol{\zeta}\|)\,.\,\boldsymbol{E}_1\end{aligned}$$

where

$$\zeta_{lk}=v_{0l}\frac{\Sigma_{s=1}^3(\beta_s+\mathrm{i}\delta_s)\xi_{sk}}{\omega-\boldsymbol{v}_0\,.\,\boldsymbol{\beta}+\mathrm{i}\boldsymbol{v}_0\,.\,\boldsymbol{\delta}}\quad\text{and}\quad\omega_\mathrm{p}^2=\frac{e\rho_0}{m\epsilon\epsilon_0},$$

and then finally to calculate the dielectric tensor $\|\boldsymbol{\epsilon}\|$ from the *total current density*, $\boldsymbol{j}+\mathrm{i}\omega\epsilon\epsilon_0\boldsymbol{E}_1=\mathrm{i}\omega\epsilon_0\|\boldsymbol{\epsilon}\|\,.\,\boldsymbol{E}_1$. It follows that

$$\|\boldsymbol{\epsilon}\|=\epsilon\left(\|U\|+\frac{\omega_\mathrm{p}^2}{\mathrm{i}\omega}(\|\boldsymbol{\xi}\|+\|\boldsymbol{\zeta}\|)\right). \tag{4.1}$$

† So that $\|\boldsymbol{\theta}\|\,.\,\boldsymbol{v}_1=(\boldsymbol{v}_1\,.\,\nabla)\boldsymbol{v}_0$.

Here ϵ is the lattice dielectric constant, and $\omega_p/2\pi$ is the plasma frequency. In the quasi-static approximation we have $\boldsymbol{E}_1 = -\nabla\phi = \mathrm{i}\boldsymbol{\beta}\phi$. Then since

$$\nabla \times \boldsymbol{H}_1 = \boldsymbol{j} + \mathrm{i}\omega\epsilon\epsilon_0 \boldsymbol{E}_1 = \mathrm{i}\omega\epsilon_0 \|\boldsymbol{\epsilon}\| \,.\, \boldsymbol{E}_1,$$

and

$$\nabla \,.\, (\nabla \times \boldsymbol{H}_1) = -\boldsymbol{\beta} \,.\, (\boldsymbol{\beta} \times \boldsymbol{H}_1) \equiv 0$$

we have

$$\mathrm{i}\omega\epsilon_0 \boldsymbol{\beta} \,.\, \|\boldsymbol{\epsilon}\| \,.\, \boldsymbol{\beta}\phi \equiv 0.$$

It follows that

$$\boldsymbol{\beta} \,.\, \|\boldsymbol{\epsilon}\| \,.\, \boldsymbol{\beta} = 0 \tag{4.2}$$

is the required dispersion relation.

5. Detailed calculation of a special case

Before inverting (3.3) we shall assume that spatial variations in $\boldsymbol{v}_0$ are small enough for $\|\boldsymbol{\theta}\|$ to be ignored. This is valid if the temperature T is large enough so that $\|\boldsymbol{g}\|$ dominates $\|\boldsymbol{\theta}\|$. This assumption proves to be a great simplification.

We also assume that $\boldsymbol{\beta}$ is normal to the direction of the magnetic field; that is, $\boldsymbol{\beta}$ lies in the (x, y) plane. By rotating the axes we can arrange for $\boldsymbol{\beta}$ to be along the x axis, $\boldsymbol{\beta} = (\beta, 0, 0)$.

We now consider the special case where $\boldsymbol{\delta} = (\delta, 0, 0)$ is parallel to the direction of propagation.

The quasistatic approximation, $\boldsymbol{\beta} \times \boldsymbol{E}_1 = 0$, now shows that $\boldsymbol{E}_1 = (E_1, 0, 0)$. Furthermore

$$\|\boldsymbol{g}\| = \begin{pmatrix} g_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where

$$g_{11} = \frac{kT}{m} \frac{\beta^2 + \delta^2}{\omega - u_0\beta + \mathrm{i}u_0\delta},$$

and where $\boldsymbol{v}_0 = (u_0, v_{0y}, v_{0z})$.

With these approximations the dispersion relation (4.2) becomes $\epsilon_{11} = 0$, that is,

$$1 + \frac{\omega_p^2}{\mathrm{i}\omega}(\xi_{11} + \zeta_{11}) = 0.$$

On writing (3.3) out in full and solving for $\boldsymbol{v}_1$ we get, on comparing with (3.4):

$$\xi_{11} = \mathrm{i}(\omega - u_0\beta - \mathrm{i}\nu)\left\{\omega_c^2 - (\omega - u_0\beta - \mathrm{i}\nu)\left(\omega - u_0\beta - \mathrm{i}\nu - \frac{kT}{m}\frac{\beta^2 + \delta^2}{\omega - u_0\beta + \mathrm{i}v_0\delta}\right)\right\}^{-1}.$$

Since

$$\zeta_{11} = \frac{u_0(\beta + \mathrm{i}\delta)\xi_{11}}{\omega - u_0\beta + \mathrm{i}u_0\delta},$$

the dispersion equation is, after re-arranging

$$\omega(\omega-u_0\beta+\mathrm{i}u_0\delta)(\omega-u_0\beta-\mathrm{i}\nu)^2-\omega_c^2(\omega-u_0\beta+\mathrm{i}u_0\delta)\omega-\omega_p^2(\omega+2\mathrm{i}u_0\delta)(\omega-u_0\beta-\mathrm{i}\nu)$$
$$-(\beta^2+\delta^2)(\omega-u_0\beta-\mathrm{i}\nu)\omega\frac{kT}{m}=0. \quad (5.1)$$

6. Application of Briggs' stability criteria

To apply these criteria it is necessary to examine the behaviour of the complex roots of the dispersion relation (5.1), which is a quartic in ω and a cubic in β with complex coefficients. One of us (DMS) developed a computer program (the details of which are given in Short 1972) which enables root loci to be displayed graphically with the incremental density gradient δ as parameter.

Figures 2–5 give some of the results, with $\nu = \omega_p = 10^{13}\,\mathrm{s}^{-1}$, $\omega_c = 10^{10}\,\mathrm{s}^{-1}$, $u_0 = 10^6\,\mathrm{m\,s}^{-1}$, $kT/m = 10^9\,\mathrm{m}^2\,\mathrm{s}^{-2}$.

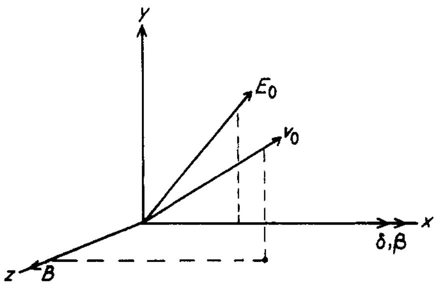

Figure 1. Orientation of the axes.

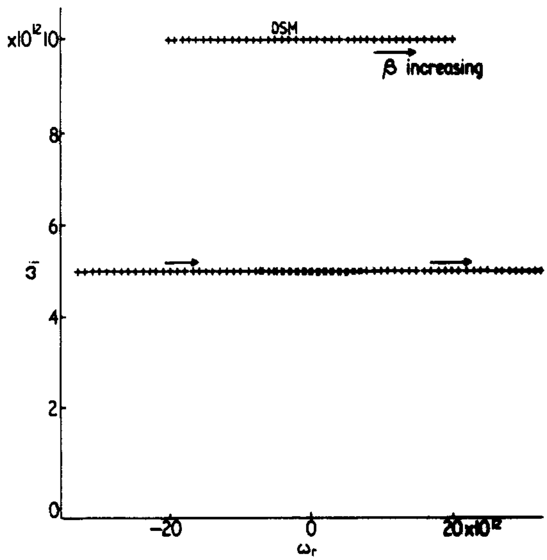

Figure 2. Roots in ω plane for β real with $\delta = 0$.

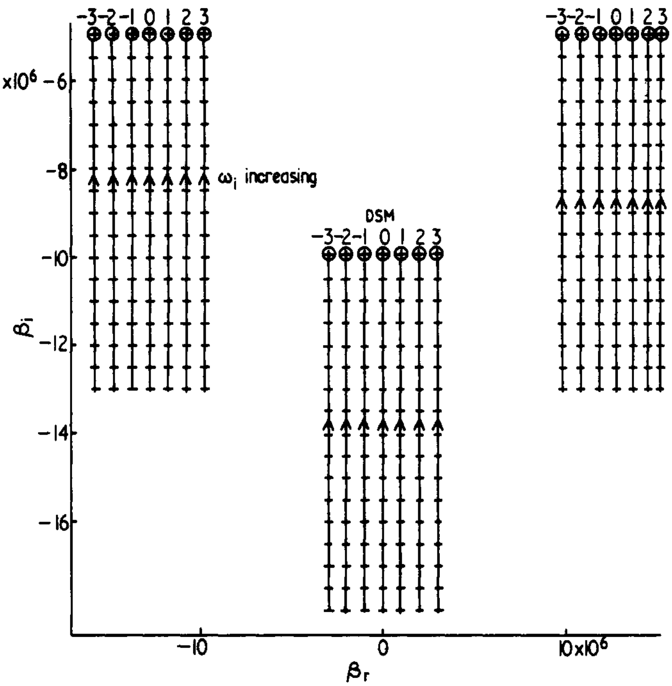

Figure 3. Roots in β plane for complex ω with $\delta = 0$.

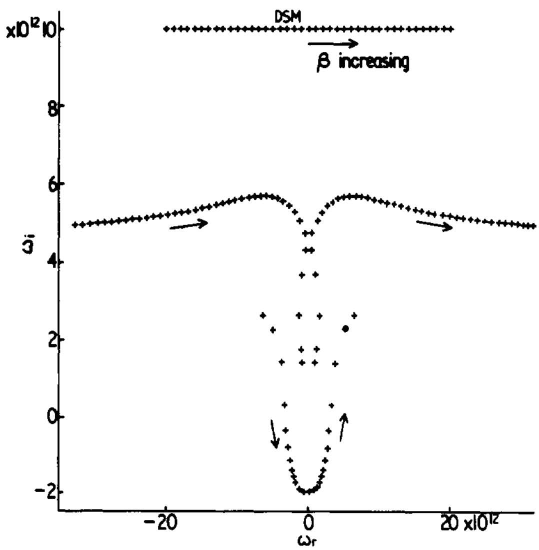

Figure 4. Roots in ω plane for β real with $\delta = v/10u_0$.

When $\delta = 0$ equation (5.1) becomes a cubic in ω and β multiplied by ω as a factor. Accordingly one of the root loci becomes the point $\omega_r = 0$ $\omega_i = 0$†, as shown in figure 2. The system is expected to be stable when $\delta = 0$, and figures 2 and 3 show that $\omega_i \geqslant 0$, $\beta_i < 0$, which confirms the stability within the ranges shown.

† Where $\omega = \omega_r + i\omega_i$. Also $\beta = \beta_r + i\beta_i$.

Figures 4 and 5 are the corresponding root loci when $\delta = v/10u_0$. Figure 4 shows a root locus with $\omega_i < 0$ for a small frequency $\omega_r/2\pi$ indicating an instability; however figure 5 shows that there is no absolute instability since the roots merging in the saddle point at S both come from the lower half-plane of β (see Briggs 1964). It is clear, however, that a convective instability exists since $\beta_i > 0$ when $\omega_i = 0$, and β_i changes sign when $\omega_i \to -\infty$ (Briggs 1964), and accordingly wave amplification can be expected provided δ is large enough.

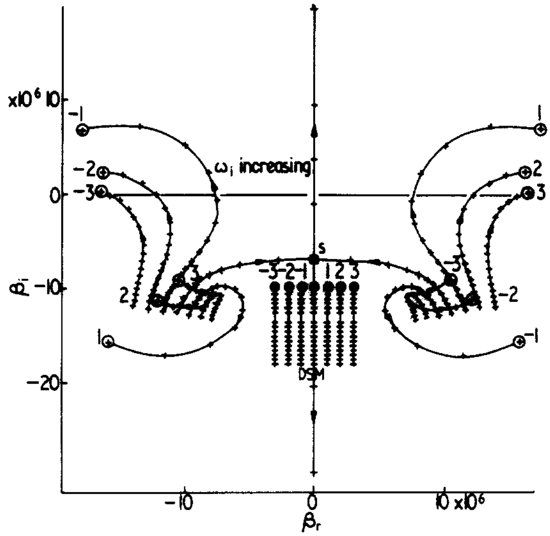

Figure 5. Roots in β plane for complex ω with $\delta = v/10u_0$.

7. Interpretation of results

Probably the best way to get a physical picture of the system is to write the equations in coupled mode form (Louisell 1960). To do this we return to the case studied in § 5.

We have $j_x + \mathrm{i}\omega\epsilon\epsilon_0 E_1 = \mathrm{i}\omega\epsilon_0\epsilon_{11}E_1 = 0$, because $\epsilon_{11} = 0$. We also have

$$j_x = \epsilon\epsilon_0\omega_p^2(\xi_{11}+\zeta_{11})E_1$$

$$= \epsilon\epsilon_0\omega_p^2\frac{\omega+2\mathrm{i}u_0\delta}{\omega-u_0\beta+\mathrm{i}u_0\delta}\xi_{11}E_1$$

from § 4. Thus the system to be normalized is

$$j_x = -\mathrm{i}\omega\epsilon\epsilon_0 E_1 \tag{7.1}$$

$$E_1 = \frac{\omega-u_0\beta+\mathrm{i}u_0\delta}{\xi_{11}(\omega+2\mathrm{i}u_0\delta)\omega_p^2\epsilon\epsilon_0}j_x. \tag{7.2}$$

Equation (7.2) can be written as $j_x = \sigma E_1$, where the complex 'conductivity' σ is a function of δ. We now write

$$\sigma(\delta) = \sigma(0)+\delta\left(\frac{\partial\sigma}{\partial\delta}\right)_{\delta=0} + \dots$$

$$= \sigma_1+\sigma_2(\delta), \qquad \text{say, where } \sigma_2(0) = 0.$$

We have seen that when $\delta = 0$ no unstable behaviour takes place, and so we expect to find uncoupled normal modes in this case.

Therefore define $a_+ \triangleq Aj_x + BE_1$, $a_- \triangleq Aj_x - BE_1$, where A and B are to be determined, and substitute into

$$E_1 = \frac{\mathrm{i}}{\omega\epsilon\epsilon_0} j_x$$

$$j_x = \sigma_1 E_1 .$$

The result is: if $B^2 = -\mathrm{i}\sigma_1\omega\epsilon\epsilon_0 A^2$ then

$$\left.\begin{aligned}(\sigma_1 A - B)a_+ &= 0\\ (\sigma_1 A + B)a_- &= 0\end{aligned}\right\}. \tag{7.3}$$

We thus have two uncoupled normal modes a_+ and a_- with dispersion relations

$$\left.\begin{aligned}\sigma_1 - \mathrm{i}\sqrt{\mathrm{i}\sigma_1\omega\epsilon\epsilon_0} &= 0\\ \sigma_1 + \mathrm{i}\sqrt{\mathrm{i}\sigma_1\omega\epsilon\epsilon_0} &= 0\end{aligned}\right\} \tag{7.4}$$

respectively†. We note in passing that

$$\sigma_1 = \frac{\mathrm{i}\epsilon\epsilon_0\omega_{\mathrm{p}}^2\omega}{\omega_{\mathrm{c}}^2 - (\omega - u_0\beta)^2}$$

when $T = \nu = 0$. Thus for the a_+ mode

$$\frac{\epsilon\epsilon_0\omega_{\mathrm{p}}^2\omega}{\omega_{\mathrm{c}}^2 - (\omega - u_0\beta)^2} = +\left(\frac{\omega^2\omega_{\mathrm{p}}^2\epsilon^2\epsilon_0^2}{(\omega - u_0\beta)^2 - \omega_{\mathrm{c}}^2}\right)^{1/2},$$

$$\omega_{\mathrm{p}}^2 = -(+)^2\{\omega_{\mathrm{c}}^2 - (\omega - u_0\beta)^2\}$$

$$\omega_{\mathrm{p}}^2 + \omega_{\mathrm{c}}^2 = (+)^2(\omega - u_0\beta)^2$$

$$\omega = u_0\beta + \sqrt{\omega_{\mathrm{p}}^2 + \omega_{\mathrm{c}}^2}.$$

$$\text{The phase velocity} = u_0 + \frac{\sqrt{\omega_{\mathrm{p}}^2 + \omega_{\mathrm{c}}^2}}{\beta}.$$

$$> u_0 \qquad \text{when } \beta > 0.$$

Again for the a_- mode at zero temperature and with no collisions

$$\omega_{\mathrm{p}}^2 + \omega_{\mathrm{c}}^2 = (-)^2(\omega - u_0\beta)^2$$

or

$$\omega = u_0\beta - \sqrt{\omega_{\mathrm{p}}^2 + \omega_{\mathrm{c}}^2}.$$

In this case the phase velocity is $u_0 - \sqrt{\omega_{\mathrm{p}}^2 + \omega_{\mathrm{c}}^2}/\beta < u_0$ always ($\omega_{\mathrm{r}} > 0$). Thus a_+ is a 'fast' mode and a_- a 'slow' mode. These modes reduce to the Hahn–Ramo modes when $\omega_{\mathrm{c}} = 0$.

On the other hand if $T \neq 0$ but $\nu = \omega_{\mathrm{c}} = 0$ we get

$$\sigma_1 = -\frac{\mathrm{i}\omega\omega_{\mathrm{p}}^2\epsilon\epsilon_0}{(\omega - u_0\beta)^2 - (kT/m)\beta^2},$$

† a_+ and a_- are calculated on the assumption that $\sigma_1 \neq 0$. The damped almost synchronous mode (DSM) appearing in figures 2–5 is a result of choosing a rather small value for $\omega_{\mathrm{c}}/\omega_{\mathrm{p}}$, for then $\omega - u_0\beta - \mathrm{i}\nu$ is almost a factor of the dispersion relation (5.1), and this implies that $\sigma_1 = 0$ gives a mode of the system.

and so the dispersion relations (7.4) combine to give

$$(\omega - u_0\beta)^2 - \frac{kT}{m}\beta^2 = \omega_p^2. \tag{7.5}$$

The temperature is held strictly constant in this calculation, whereas in general temperature waves should also be allowed for. To do this put $T = T_0 + T_1$, and replace the first-order pressure term

$$\frac{kT}{m}\left(\frac{\nabla\rho}{\rho} - \frac{\nabla\rho_0}{\rho_0}\right) \qquad \text{by} \qquad \frac{k}{m}\left(\frac{\nabla T\rho}{\rho} - \frac{\nabla T_0\rho_0}{\rho_0}\right)$$

in the first-order force equation of § 2. As a consequence a term

$$\frac{ikT_0}{m}(\boldsymbol{\beta} + i\boldsymbol{\delta})\frac{T_1}{T_0}$$

appears on the right-hand side of the first of equations (3.3). Also a suffix '0' must be attached to T in the expression for g_{kl}.

The heat flux equation is now needed to eliminate T_1/T_0 as was done by Boyd *et al* (1972). If the system obeys a quasi-adiabatic law for example (Newton 1963) with one degree of freedom then $T_1/T_0 = 2\rho_1/\rho_0$.

In this case (7.5) is replaced by

$$(\omega - u_0\beta)^2 - \frac{3kT_0}{m}\beta^2 = \omega_p^2$$

which, when $u_0 = 0$, leads to the well-known relation of Bohm and Gross, or Vlasov.

In what follows we shall stick to our assumption of strictly constant temperature.

8. Kinetic power flow by means of the carriers

By Chu's kinetic power theorem (Steele and Vural 1969) the power flow (in W m^{-2}) is

$$P = -\tfrac{1}{2}\,\mathrm{Re}(Vj_x^*),$$

where

$$\frac{e}{m}V \triangleq u_0 v_{1x}$$

$$= \frac{e}{m}u_0\xi_{11}E_1 \qquad \text{(by (3.4))}.$$

Accordingly

$$P = -\tfrac{1}{2}u_0\,\mathrm{Re}(\xi_{11}E_1 j_x^*).$$

But

$$\xi_{11}E_1 j_x^* = \xi_{11}\frac{|a_+|^2 - |a_-|^2 - a_-a_+^* + a_-^*a_+}{4A^*B}$$

and

$$\xi_{11}^*E_1^*j_x = \xi_{11}^*\frac{|a_+|^2 - |a_-|^2 + a_-a_+^* - a_-^*a_+}{4AB^*},$$

whence

$$P = -\frac{u_0}{16}\left\{\left(\frac{\xi_{11}}{A^*B}+\frac{\xi_{11}^*}{AB^*}\right)(|a_+|^2-|a_-|^2)+\left(\frac{\xi_{11}}{A^*B}-\frac{\xi_{11}^*}{AB^*}\right)(a_-^*a_+-a_-a_+^*)\right\}.$$

We have already fixed the ratio A/B in § 7. We now further restrict A and B to be such that the energy density of the two modes is $|a_+|^2$ and $|a_-|^2$ respectively. Then if W is the energy density of the system we have

$$\begin{aligned} W = |a_+|^2+|a_-|^2 &= 2\{|A|^2|j_x|^2+|B|^2|E_1|^2\} \\ &= 2\{|A|^2|\sigma_1|^2+|B|^2\}\,|E_1|^2. \end{aligned}$$

From (7.3)

$$B = \pm\sigma_1 A, \tag{8.1}$$

and it follows immediately that

$$W = 4|A|^2|\sigma_1|^2|E_1|^2. \tag{8.2}$$

Clearly

$$\begin{aligned} \frac{\xi_{11}}{A^*B}+\frac{\xi_{11}^*}{AB^*} &= \frac{2}{|A|^2|\sigma_1|^2}\,\mathrm{Re}(\xi_{11}\sigma_1^*) \\ \frac{\xi_{11}}{A^*B}-\frac{\xi_{11}^*}{AB^*} &= \frac{2\mathrm{i}}{|A|^2|\sigma_1|^2}\,\mathrm{Im}(\xi_{11}\sigma_1^*), \end{aligned}$$

using (8.1) and (8.2).

The kinetic power flow is therefore

$$P = -\frac{1}{8}\frac{u_0}{|A|^2|\sigma_1|^2}\{(|a_+|^2-|a_-|^2)\,\mathrm{Re}(\xi_{11}\sigma_1^*)+\mathrm{i}(a_-^*a_+-a_-a_+^*)\,\mathrm{Im}(\xi_{11}\sigma_1^*)\}.$$

When only the mode a_- is present, so that $a_+ = 0$,

$$P = P_- = \frac{1}{8}\frac{u_0}{|A|^2|\sigma_1|^2}|a_-|^2\,\mathrm{Re}(\xi_{11}\sigma_1^*).$$

If $a_- = 0$,

$$P = P_+ = -\frac{1}{8}\frac{u_0}{|A|^2|\sigma_1|^2}|a_+|^2\,\mathrm{Re}(\xi_{11}\sigma_1^*).$$

It follows that

$$P_-P_+ = -\frac{1}{64}\frac{u_0^2}{|A|^4|\sigma_1|^4}|a_+|^2|a_-|^2\{\mathrm{Re}(\xi_{11}\sigma_1^*)\}^2$$

which is negative, so that when one mode is *passive* (positive energy carrying) then the other is *active* (negative energy carrying).

This is an unstable situation since if there is coupling such that energy is removed from the active wave and added to the passive wave both amplitudes grow.

9. Mode coupling

When $\delta \neq 0$ our equations become

$$E_1 = \frac{\mathrm{i}}{\omega\epsilon\epsilon_0} j_x$$

$$j_x = \sigma_1 E_1 + \sigma_2 E_1 .$$

Equations (7.3) are now replaced by

$$(\sigma_1 A - B)a_+ = -\tfrac{1}{2}\sigma_2 A(a_+ - a_-)$$

$$(\sigma_1 A + B)a_- = \tfrac{1}{2}\sigma_2 A(a_+ - a_-).$$

This indicates that a_+ and a_- are strongly coupled, the mutual coupling being the same in both directions between the modes. The dispersion relations are

$$B = A\sqrt{(\sigma_1 - \tfrac{1}{2}\sigma_2)(\sigma_1 + \tfrac{3}{2}\sigma_2)}$$

and

$$B = -A\sqrt{(\sigma_1 - \tfrac{1}{2}\sigma_2)(\sigma_1 + \tfrac{3}{2}\sigma_2)}$$

respectively, with $B^2 = \mathrm{i}\sigma_1\omega\epsilon\epsilon_0 A^2$ as before.

10. Conclusion

We have shown that the unstable behaviour of the particular case analysed in §§ 5 and 6 is due to coupling between travelling-wave modes: coupling which is present only when density gradients are present.

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